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Error Bounds for Lagrange Interpolation

ALEKSEI SHADRIN

Computing Center, Siberian Branch, Russian Academy of Sciences, Novosibirsk 630090, Russia

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In this paper we study the quantities

$$L_{m,k}(\Delta, x) = \sup_{\|f^{(m)}\| \le 1} \left| f^{(k)}(x) - l_{m-1,\Delta}^{(k)}(f, x) \right|,$$

$$L_{m,k}(\Delta) = \sup_{x \in [a,b]} L_{m,k}(\Delta, x),$$

which define error bounds for the approximation of functions $f \in W_\infty^m[a,b]$ by the interpolating Lagrange polynomials $l_{m-1,\Delta}(f)$ of degree $m-1$, constructed on the given mesh of interpolating nodes

$$\Delta = \Delta_m = \{a \le t_1 < \cdots < t_m \le b\}.$$

Set

$$\omega_\Delta(x) = \prod_{i=1}^{m} (x - t_i).$$

It is clear that

$$L_{m,k}(\Delta, x) \ge \frac{1}{m!} \left| \omega_\Delta^{(k)}(x) \right|, \qquad L_{m,k}(\Delta) \ge \frac{1}{m!} \left\| \omega_\Delta^{(k)}(\cdot) \right\|.$$

Our main result is

THEOREM 1. *For all m and k $(0 \le k \le m-1)$, and for any mesh Δ of the interpolating nodes $\{t_i\}_1^m$*

$$L_{m,k}(\Delta) = \frac{1}{m!} \left\| \omega_\Delta^{(k)}(\cdot) \right\|.$$

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1. Introduction

In this paper we study the quantities

$$L_{m,k}(\Delta, x) = \sup_{\|f^{(m)}\| \le 1} \left| f^{(k)}(x) - l^{(k)}_{m-1,\Delta}(f, x) \right|, \tag{1.1}$$

$$L_{m,k}(\Delta) = \sup_{x \in [a,b]} L_{m,k}(\Delta, x),$$

which define error bounds for the approximation of functions $f \in W^m_\infty[a, b]$ by the interpolating Lagrange polynomials $l_{m-1,\Delta}(f)$ of degree $m - 1$, constructed on the given mesh of interpolating nodes

$$\Delta = \Delta_m = \{a \le t_1 < \cdots < t_m \le b\}.$$

Set

$$\omega_\Delta(x) = \prod_{i=1}^{m} (x - t_i).$$

It is clear that

$$L_{m,k}(\Delta, x) \ge \frac{1}{m!} \left| \omega^{(k)}_\Delta(x) \right|, \qquad L_{m,k}(\Delta) \ge \frac{1}{m!} \left\| \omega^{(k)}_\Delta(\cdot) \right\|.$$

Our main result is

Theorem 1. *For all m and k $(0 \le k \le m-1)$, and for any mesh Δ of the interpolating nodes $\{t_i\}_1^m$*

$$L_{m,k}(\Delta) = \frac{1}{m!} \left\| \omega^{(k)}_\Delta(\cdot) \right\|. \tag{1.2}$$

This theorem is an immediate consequence of the two following statements concerning the value of the point-wise deviation $L_{m,k}(\Delta, x)$. Their formulations include two subsets of the interval $[a, b]$

$$I_{m,k}(\Delta) = \bigcup_{j=0}^{m-k} [\alpha_j, \beta_j], \qquad J_{m,k}(\Delta) = \bigcup_{j=0}^{m-k-1} (\beta_j, \alpha_{j+1}),$$

$$I_{m,k}(\Delta) \cap J_{m,k}(\Delta) = \varnothing, \qquad I_{m,k}(\Delta) \cup J_{m,k}(\Delta) = [a, b],$$

which will be specified later.

THEOREM A. *For all m and k, and for any mesh Δ*

$$L_{m,k}(\Delta, x) = \frac{1}{m!}\left|\omega_\Delta^{(k)}(x)\right|, \qquad x \in I_{m,k}(\Delta).$$

THEOREM 2. *For all m and k, and for any mesh Δ the function $L_{m,k}(\Delta, x)$ has on each interval $(\beta_j, \alpha_{j+1}) \subset J_{m,k}(\Delta)$ at most one extremum, which in this case is the minimum; thus,*

$$L_{m,k}(\Delta, x) < \max\left\{\frac{1}{m!}\left|\omega_\Delta^{(k)}(\beta_j)\right|, \frac{1}{m!}\left|\omega_\Delta^{(k)}(\alpha_{j+1})\right|\right\},$$
$$x \in (\beta_j, \alpha_{j+1}) \subset J_{m,k}(\Delta). \tag{1.3}$$

Theorem A was established by H. Kallioniemi [2]. In [3] he made a conjecture that Theorem 2 is valid and gave some conditions on the mesh Δ, providing equality (1.2).

In [6], we proved Theorem 2 for $k = m - 1$. If $k = 0$, then $J_{m,k}(\Delta) = \varnothing$. In this paper we give a proof of Theorem 2 for arbitrary k, $1 \le k \le m - 2$.

The idea of the proof follows that of V. A. Markov [5, p. 88]. Theorems A and 2 are similar to his results concerning the value

$$N_{m,k}(x) = \sup_{\|p_{m-1}\| \le 1} |p_{m-1}^{(k)}(x)|, \tag{1.4}$$

which defines the norm of the functional of the kth derivative at the point $x \in [a, b]$ on the class of algebraic polynomials of degree $m - 1$. (See [1] for a condensed original proof of V. A. Markov's inequality.)

Along with problems (1.1)–(1.2) it is natural to consider the problem of finding optimal formulas for the Lagrange interpolation and calculating the value

$$L_{m,k} = \inf_{\Delta \subset [a,b]} L_{m,k}(\Delta).$$

Theorem 1 reduces this problem to a minimization problem on the class of polynomials.

COROLLARY. *For all m and k $(0 \le k \le m - 1)$*

$$L_{m,k} = \inf_{\Delta \subset [a,b]} \frac{1}{m!}\left\|\omega_\Delta^{(k)}(\cdot)\right\|.$$

Characterization of an extremal polynomial requires special considerations. Here, without going into details, we mention the papers [3, 4, 6], where the cases $k = 0, 1$, and $k \ge [m/2]$ were considered.

2. Preliminaries

For

$$\Delta = \Delta_m = \{a \le t_1 < \cdots < t_m \le b\}$$

set

$$\omega(x) = \omega_\Delta(x) = \prod_{i=1}^{m}(x - t_i),$$

$$\omega_i(x) = \omega_\Delta(x)/(x - t_i), \qquad i = \overline{1, m};$$

$$l_i(x) = \omega_i(x)/\omega_i(t_i), \qquad i = \overline{1, m};$$

$$l_{m-1,\Delta}(f, x) = \sum_{1}^{m} l_i(x) f(t_i).$$

Further, denote

$$z_+ = \max(0, z), \qquad z_- = \max(0, -z), \qquad z_\pm^l = z^l \cdot \operatorname{sign} z$$

and introduce the functions

$$S(x,\theta) = \frac{1}{m!}(x-\theta)_\pm^m - l_{m-1,\Delta}\left(\frac{1}{m!}(\cdot - \theta)_+^m, x\right) \tag{2.1}$$

$$= \frac{1}{m!}(x-\theta)_\pm^m - \frac{1}{m!}\sum_{i=1}^{m}\frac{\omega_i(x)}{\omega_i(t_i)}(t_i - \theta)_+^m, \tag{2.2}$$

$$2 \cdot B(x,\theta) = \frac{1}{(m-1)!}(x-\theta)_\pm^{m-1} - l_{m-1,\Delta}\left(\frac{1}{(m-1)!}(\cdot - \theta)_\pm^{m-1}, x\right) \tag{2.3}$$

$$= \frac{1}{(m-1)!}(x-\theta)_\pm^{m-1} - \frac{1}{(m-1)!}\sum_{i=1}^{m}\frac{\omega_i(x)}{\omega_i(t_i)}(t_i - \theta)_+^{m-1}. \tag{2.4}$$

Note that based on the relations

$$z_\pm^l = (-1)^{l-1} z_-^l + z_+^l, \qquad z^l = (-1)^l z_-^l + z_+^l$$

$$z^l(\cdot) - l_{m-1,\Delta}(z^l, \cdot) \equiv 0, \qquad l = \overline{0, m-1};$$

we can represent $B(x, \theta)$ in the form

$$B(x,\theta) = \frac{1}{(m-1)!}(x-\theta)_+^{m-1} - \frac{1}{(m-1)!}\sum_{i=1}^{m} \frac{\omega_i(x)}{\omega_i(t_i)}(t_i-\theta)_+^{m-1}. \tag{2.5}$$

Finally, set

$$S_l(x,\theta) = \frac{\partial^l}{\partial x^l} S(x,\theta), \qquad l = \overline{0, m};$$

$$B_l(x,\theta) = \frac{\partial^l}{\partial x^l} B(x,\theta), \qquad l = \overline{0, m-1}.$$

LEMMA A. *Let*

$$\omega_m^{(k)}(x) = c\prod_j (x-\alpha_j), \qquad \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1-k};$$

$$\omega_1^{(k)}(x) = c\prod_j (x-\beta_j), \qquad \beta_1 < \beta_2 < \cdots < \beta_{m-1-k};$$

$$\alpha_0 = a, \qquad \beta_0 = t_1, \qquad \alpha_{m-k} = t_m, \qquad \beta_{m-k} = b.$$

Then

$$\alpha_j \le \beta_j \le \alpha_{j+1} \le \beta_{j+1}$$

and on each interval (β_j, α_{j+1}), $j = \overline{0, m-1-k}$, *the equalities*

$$\operatorname{sign} \omega_i^{(k)}(\cdot) = \operatorname{sign} \omega_\Delta^{(k+1)}(\cdot), \qquad i = \overline{1, m}.$$

are valid.

The subsets $I_{m,k}(\Delta)$ and $J_{m,k}(\Delta)$ from Theorems A and 2 are defined as follows

$$I_{m,k}(\Delta) = \bigcup_{j=0}^{m-k} [\alpha_j, \beta_j],$$

$$J_{m,k}(\Delta) = \bigcup_{j=0}^{m-k-1} (\beta_j, \alpha_{j+1}) \subset (t_1, t_m).$$

Moreover,

$$I_{m,k}(\Delta) = [a,b], \qquad k = 0;$$
$$J_{m,k}(\Delta) = (t_1, t_m), \qquad k = m-1;$$
$$\operatorname{mes} J_{m,k}(\Delta) = \frac{k}{m-1}(t_m - t_1), \qquad 0 \le k \le m-1. \tag{2.6}$$

LEMMA B. *For* $f \in W_\infty^m[a,b]$

$$f^{(k)}(x) - l_{m-1,\Delta}^{(k)}(f,x) = \int_a^b B_k(x,\theta) f^{(m)}(\theta)\, d\theta.$$

LEMMA C. *The kernel* $B_k(x,\theta) = B_{k,\Delta}(x,\theta)$ *has the following properties*:

(i) *if* $x \in I_{m,k}(\Delta)$, *then the function* $B_k(x,\cdot)$ *does not change its sign on the interval* $[a,b]$;

(ii) *if* $x \in J_{m,k}(\Delta)$, *then* supp $B_k(x,\cdot) = (t_1,t_m)$ *and on the interval* (t_1,t_m) *the function* $B_k(x,\cdot)$ *has exactly one zero* $\theta = \theta_x$, *this zero is a single one, and, therefore*, $B_k(x,\cdot)$ *changes its sign exactly once*;

(iii) *if* $x \in (t_1,t_m)$, *then*

$$\left.\frac{\partial^l}{\partial\theta^l} B_k(x,\theta)\right|_{\theta = t_1, t_m} = 0, \qquad l = \overline{0, m-2}.$$

Lemmas A–C can be found in [6] (see also [2–3]). Lemma A is derived from a theorem by V. A. Markov which states that if the roots of two polynomials are real and interlace, then the same is true for the roots of their derivatives [5, Corollary from Lemma 3]. Relation (2.6) is established in [3] following the idea from [1].

Theorem A follows from Lemma B and item (i) of Lemma C. From Lemma B and item (ii) of Lemma C one can conclude that for $x \in J_{m,k}(\Delta)$ the extremal function, which attains the supremum in (1.1), satisfies the relation

$$f_*^{(m)}(x) = \pm \operatorname{sign}(x - \theta_x).$$

Hence, with the aid of the equalities

$$\frac{\partial^m}{\partial x^m} S(x,\theta) = \operatorname{sign}(x-\theta), \qquad l_{m-1,\Delta}(S(\cdot,\theta),x) \equiv 0$$

we obtain

LEMMA D. *If* $x \in J_{m,k}(\Delta)$, *then*

$$L_{m,k}(\Delta,x) = |S_k(x,\theta_x)|,$$

with θ_x the unique point from (t_1, t_m) such that

$$B_k(x, \theta_x) = 0.$$

3. Proof of Theorem 2

We will prove the following statement.

Theorem 2'. *Let $1 \leq k \leq m-2$, $x \in (\beta_j, \alpha_{j+1}) \subset J_{m,k}(\Delta)$, and*

$$M_k(x) = S_k(x, \theta_x),$$

where θ_x is such that

$$|S_k(x, \theta_x)| = L_{m,k}(\Delta, x).$$

If at any point $z \in (\beta_j, \alpha_{j+1})$

$$M_k'(z) = 0, \tag{3.1}$$

then

$$M_k(z) \cdot M_k''(z) > 0. \tag{3.2}$$

From (3.1)–(3.2) it follows that if the extremum of the function $L_{m,k}(\Delta, x) = |M_k(x)|$ on the interval (β_j, α_{j+1}) exists, then it must be the minimum, what proves that there is at most one such extrema, and this is exactly what Theorem 2 states.

Proof of Theorem 2'. Comparing (2.2) and (2.4), we see that

$$\frac{\partial}{\partial\theta} S_l(x, \theta) = -2 \cdot B_l(x, \theta), \qquad l = \overline{0, m-1}. \tag{3.3}$$

Furthermore, set

$$B_l'(x, \theta) = \frac{\partial}{\partial\theta} B_l(x, \theta), \qquad l = \overline{0, m-2}$$

So, we have

$$M_k(x) = S_k(x, \theta_x), \tag{3.4}$$

where by Lemma D

$$B_k(x, \theta_x) = 0. \tag{3.5}$$

Hence,

$$\begin{aligned} M_k'(x) &= \frac{\partial}{\partial x} S_k(x,\theta)\Big|_{\theta=\theta_x} + \frac{\partial}{\partial\theta} S_k(x,\theta)\Big|_{\theta=\theta_x} \cdot \theta'(x) \\ &= S_{k+1}(x,\theta_x) - 2\cdot B_k(x,\theta_x)\cdot\theta'(x). \end{aligned}$$

Differentiating the identity $B_k(x,\theta(x)) \equiv 0$ with respect to x, we find

$$\theta'(x) = -B_{k+1}(x,\theta_x)/B_k'(x,\theta_x),$$

and since θ_x is a single zero of the function $B_k(x,\cdot)$, we have $B_k'(x,\theta_x) \neq 0$, i.e., $|\theta'(x)| < \infty$.

Thus,

$$M_k'(x) = S_{k+1}(x,\theta_x),$$

and, therefore,

$$S_{k+1}(z,\theta_z) = 0. \tag{3.6}$$

Further,

$$\begin{aligned} M_k''(x) &= \frac{\partial}{\partial x} S_{k+1}(x,\theta)\Big|_{\theta=\theta_x} + \frac{\partial}{\partial\theta} S_{k+1}(x,\theta)\Big|_{\theta=\theta_x} \cdot \theta'(x) \\ &= S_{k+2}(x,\theta_x) - 2\cdot B_{k+1}(x,\theta_x)\cdot\theta'(x) \\ &= S_{k+2}(x,\theta_x) + 2\cdot B_{k+1}^2(x,\theta_x)/B_k'(x,\theta_x). \end{aligned}$$

Using (3.4) we finally obtain

$$M_k(x)\cdot M_k''(x) = \frac{S_k(x,\theta_x)}{B_k'(x,\theta_x)}\big(S_{k+2}(x,\theta_x)\cdot B_k'(x,\theta_x) + 2\cdot B_{k+1}^2(x,\theta_x)\big).$$

Let us show that the inequalities

$$\frac{S_k(z,\theta_z)}{B_k'(z,\theta_z)} > 0, \tag{3.7}$$

$$|B_k'(z,\theta_z)| < |B_{k+1}(z,\theta_z)|, \tag{3.8}$$

$$|S_{k+2}(z,\theta_z)| < |2\cdot B_{k+1}(z,\theta_z)|, \tag{3.9}$$

are valid, which proves (3.2).

Define the functions $p(x,\theta)$ and $q(x,\theta)$ by the identities

$$B'(x,\theta) = -B_1(x,\theta) + \tfrac{1}{2}p(x,\theta), \tag{3.10}$$

$$S_1(x,\theta) = 2 \cdot B(x,\theta) + q(x,\theta), \tag{3.11}$$

and note that for each $\theta \in \mathbb{R}$ the functions $p_\theta(\cdot) \equiv p(\cdot,\theta)$, $q_\theta(\cdot) \equiv q(\cdot,\theta)$ are algebraic polynomials of degree $m-1$. Then

$$B_k'(x,\theta_x) = -B_{k+1}(x,\theta_x) + \tfrac{1}{2}p_{\theta_x}^{(k)}(x), \tag{3.12}$$

$$S_{k+2}(x,\theta_x) = 2 \cdot B_{k+1}(x,\theta_x) + q_{\theta_x}^{(k+1)}(x). \tag{3.13}$$

Moreover, by (3.5)–(3.6)

$$q_{\theta_z}^{(k)}(z) = S_{k+1}(z,\theta_z) - 2 \cdot B_k(z,\theta_z) = 0. \tag{3.14}$$

The proofs of inequalities (3.7)–(3.9) are based on relations (3.12)–(3.14) and on the following three lemmas, which will be established below.

Lemma 1. *If $x \in J_{m,k}(\Delta)$ and $B_k(x,\theta_x) = 0$, then*

$$\operatorname{sign} S_k(x,\theta_x) = \operatorname{sign} B_k'(x,\theta_x), \tag{3.15}$$

$$\operatorname{sign} S_k(x,\theta_x) = -\operatorname{sign} \omega_\Delta^{(k+1)}(x), \tag{3.16}$$

$$\operatorname{sign} S_k(x,\theta_x) = \operatorname{sign} B_{k-1}(x,\theta_x), \tag{3.17}$$

$$\operatorname{sign} B_{k-1}(x,\theta_x) = -\operatorname{sign} B_{k+1}(x,\theta_x), \tag{3.18}$$

and for $x = z$, i.e., if $S_{k+1}(x,\theta_x) = 0$, moreover,

$$\operatorname{sign} S_k(x,\theta_x) = -\operatorname{sign} S_{k+2}(x,\theta_x). \tag{3.19}$$

Lemma 2. *If $x \in J_{m,k}(\Delta)$, then for any $\theta \in (t_1, t_m)$*

$$\operatorname{sign} p_\theta^{(k)}(x) = \operatorname{sign} \omega_\Delta^{(k+1)}(x).$$

Lemma 3. *If $\theta \in (t_1, t_m)$ and $q_\theta^{(k)}(x) = 0$, then*

$$\operatorname{sign} q_\theta^{(k+1)}(x) = -\operatorname{sign} \omega_\Delta^{(k+1)}(x).$$

From Lemma 1 there follow the equalities

$$\operatorname{sign} B_k'(z,\theta_z) = -\operatorname{sign} B_{k+1}(z,\theta_z), \tag{3.20}$$

$$\operatorname{sign} S_{k+2}(z,\theta_z) = \operatorname{sign} B_{k+1}(z,\theta_z), \tag{3.21}$$

$$\operatorname{sign} \omega_\Delta^{(k+1)}(z) = \operatorname{sign} B_{k+1}(z,\theta_z). \tag{3.22}$$

From Lemma 2 with the aid of (3.22) we derive

$$\operatorname{sign} p_{\theta_z}^{(k)}(z) = \operatorname{sign} B_{k+1}(z,\theta_z). \tag{3.23}$$

From Lemma 3, using (3.14) and with the aid of (3.22) we obtain

$$\operatorname{sign} q_{\theta_z}^{(k+1)}(z) = -\operatorname{sign} B_{k+1}(z,\theta_z). \tag{3.24}$$

Putting together (3.20) and (3.23), we have

$$\operatorname{sign} B_k'(z,\theta_z) = \operatorname{sign}\{-B_{k+1}(z,\theta_z)\} = -\operatorname{sign} p_{\theta_z}^{(k)}(z),$$

which by comparison with (3.12) proves (3.8).

Similarly, combining (3.21) and (3.24) we obtain

$$\operatorname{sign} S_{k+2}(z,\theta_z) = \operatorname{sign} B_{k+1}(z,\theta_z) = -\operatorname{sign} q_{\theta_z}^{(k+1)}(z),$$

and with the aid of (3.13) it proves (3.9).

Finally, inequality (3.7) is equal to (3.15). Theorem 2′ and, thus, Theorem 2 are proved.

As we pointed out in the introduction, there is a complete similarity between Theorems A and 2, which describe the behaviour of $L_{m,k}(\Delta, x)$, and V. A. Markov's results [5] on the function $N_{m,k}(x)$ (see Eq. (1.4)). The situation becomes different if we consider the behaviour of derivatives. V. A. Gusev [1] has shown that the function $N_{m,k}'(x)$ is continuous, while $N_{m,k}''(x)$ has discontinuities of the 1st kind. We give without proof the following statement on the function $L_{m,k}'(\Delta, x)$.

PROPOSITION. *The function $L_{m,k}'(\Delta, x)$ is continuous everywhere except at the points $\{\alpha_j, \beta_j\}_{j=1}^{m-k-1}$, where it has discontinuities of the* 1st *kind. Moreover*,

$$\begin{aligned}|L_{m,k}'(\Delta,\alpha_j-0)| &= |S_{k+1}(\alpha_j,t_{m-1})| \\ &\neq |S_{k+1}(\alpha_j,t_m)| = \frac{1}{m!}|\omega_\Delta^{(k+1)}(\alpha_j)| = |L_{m,k}'(\Delta;\alpha_j+0)|,\\ |L_{m,k}'(\Delta,\beta_j-0)| &= \frac{1}{m!}|\omega_\Delta^{(k+1)}(\beta_j)| = |S_{k+1}(\beta_j,t_1)| \\ &\neq |S_{k+1}(\beta_j,t_2)| = |L_{m,k}'(\Delta,\beta_j+0)|.\end{aligned}$$

The points $\{\alpha_j, \beta_j\}_{j=1}^{m-k-1}$ appear to be the breakpoints of $L'_{m,k}(\Delta, x)$ due to the fact [6] that

$$\operatorname{supp} B_k(x, \cdot) = (t_1, t_{m-1}), \qquad x \in \{\alpha_j\}_{j=1}^{m-k-1};$$

$$\operatorname{supp} B_k(x, \cdot) = (t_2, t_m), \qquad x \in \{\beta_j\}_{j=1}^{m-k-1};$$

while

$$\operatorname{supp} B_k(x, \cdot) = (t_1, t_m), \qquad x \in (t_1, t_m) \setminus \{\alpha_j, \beta_j\}_{j=1}^{m-k-1}.$$

Let us emphasize that at the points $\beta_0 = t_1$ and $\alpha_{m-k} = t_m$ the function $L'_{m,k}(\Delta, x)$ is continuous.

4. Auxiliary Properties of the Functions $B(x, \theta)$ and $S(x, \theta)$

Lemma 4. *For arbitrary mesh Δ_m, and for any $x, \theta \in \mathbb{R}$*

$$S(x,\theta) = 2 \cdot \frac{1}{m}(x - \theta) B(x,\theta) + c(\theta) \frac{1}{m!} \omega(x), \tag{4.1}$$

where

$$c(\theta) = \sum_{i=1}^{m} \frac{(t_i - \theta)_\pm^{m-1}}{\omega_i(t_i)} = \begin{cases} 1, & \theta \le t_1; \\ c_\theta \in (-1, 1), & t_1 < \theta < t_m; \\ -1, & t_m \le \theta. \end{cases} \tag{4.2}$$

Proof. By (2.2), (2.4)

$$S(t_i, \theta) = 0, \qquad B(t_i, \theta) = 0, \qquad i = \overline{1, m};$$

and since the difference

$$S(x,\theta) - 2 \cdot \frac{1}{m}(x - \theta) B(x,\theta)$$
$$= \frac{1}{m!}(x - \theta) \sum_{i=1}^{m} \frac{\omega_i(x)}{\omega_i(t_i)} (t_i - \theta)_\pm^{m-1} - \frac{1}{m!} \sum_{i=1}^{m} \frac{\omega_i(x)}{\omega_i(t_i)} (t_i - \theta)_\pm^{m}$$

is a polynomial of degree m with respect to x, we obtain

$$S(x,\theta) = 2 \cdot \frac{1}{m}(x - \theta) B(x,\theta) + c(\theta) \frac{1}{m!} \omega(x),$$

with

$$c(\theta) = \sum_{i=1}^{m} \frac{(t_i - \theta)_{\pm}^{m-1}}{\omega_i(t_i)}.$$

To prove the right-hand side of (4.2) let us introduce the classical B-spline $b(t)$ of degree $m-2$, defined on the mesh Δ_m by the formulas

$$b(t) = (m-1) \sum_{i=1}^{m} \frac{(t_i - t)_{+}^{m-2}}{\omega'_{\Delta}(t_i)}$$
$$= (-1)^{m-1}(m-1) \sum_{i=1}^{m} \frac{(t - t_i)_{+}^{m-2}}{\omega'_{\Delta}(t_i)},$$

with the properties

$$\operatorname{supp} b(\cdot) = (t_1, t_m), \qquad b(\cdot) \geq 0, \qquad \int_{t_1}^{t_m} b(t)\, dt = 1.$$

From the equalities

$$(m-1) \int_{\theta}^{t_m} (t_i - t)_{+}^{m-2}\, dt = (t_i - \theta)_{+}^{m-1},$$

$$(m-1) \int_{t_1}^{\theta} (t - t_i)_{+}^{m-2}\, dt = (\theta - t_i)_{+}^{m-1} = (t_i - \theta)_{-}^{m-1},$$

$$(t_i - \theta)_{\pm}^{m-1} = (-1)^{m-2}(t_i - \theta)_{-}^{m-1} + (t_i - \theta)_{+}^{m-1}$$

we conclude that

$$c(\theta) = \int_{\theta}^{t_m} b(t)\, dt - \int_{t_1}^{\theta} b(t)\, dt,$$

and the right-hand side equality in (4.2) follows now from the properties of B-splines.

LEMMA 5. *Let* $\theta \in (t_1, t_m)$, $k = \overline{1, m-2}$. *If*

$$B_k(y, \theta) = 0, \qquad y \in (t_1, t_m);$$

then

$$B_{k-1}(y, \theta) \cdot B_{k+1}(y, \theta) < 0. \tag{4.3}$$

LEMMA 6. *Let* $\theta \in (t_1, t_m)$, $k = \overline{1, m-2}$. *If*

$$S_{k+1}(y, \theta) = 0, \qquad y \in (t_1, t_m);$$

then

$$S_k(y, \theta) \cdot S_{k+2}(y, \theta) \le 0. \tag{4.4}$$

Proofs of Lemmas 5 *and* 6. Each of the functions $B(\cdot, \theta)$ and $S(\cdot, \theta)$ has m zeroes at the points $x = t_i$, $i = \overline{1, m}$. Moreover, their higher derivatives

$$B_{m-1}(\cdot, \theta) = \tfrac{1}{2}\operatorname{sign}(\cdot - \theta) - \tfrac{1}{2}c(\theta),$$
$$S_m(\cdot, \theta) = \operatorname{sign}(\cdot - \theta)$$

have exactly one change of sign on (t_1, t_m). Therefore, by Rolle's Theorem, for the number ν of zeroes of the functions $B_l(\cdot, \theta)$ and $S_l(\cdot, \theta)$ on the interval $[t_1, t_m]$ we have

$$\nu[B_l(\cdot, \theta)] = m - l, \qquad l = \overline{0, m-2}; \tag{4.5}$$

$$m - l \le \nu[S_l(\cdot, \theta)] \le m + 1 - l, \qquad l = \overline{0, m-1}. \tag{4.6}$$

If, for instance, (4.4) does not hold, then the interval linking the separated zeroes of the function $S_k(\cdot, \theta)$, closest to the point y, contains 3 zeroes of $S_{k+1}(\cdot, \theta)$, and if we add to this number $m - (k+2)$ zeroes of $S_{k+1}(\cdot, \theta)$, which are contained between the other zeroes of $S_k(\cdot, \theta)$, we will come to a contradiction with (4.6). The proof of (4.3) is obtained in the same manner.

Remark. As it was pointed out by one of the referees the precise statement of (4.4) is a strong inequality, and also the sharp statement of (4.6) is

$$\nu[S_l(\cdot, \theta)] = m + 1 - l, \qquad l = \overline{0, m-1}.$$

But such a refinement will not be required in the following considerations.

LEMMA 7. *Let* $\theta \in (t_1, t_m)$. *Then*

$$\operatorname{sign} B(\cdot, \theta) = \operatorname{sign} \omega(\cdot).$$

Proof. By (4.5) the function $B(\cdot, \theta)$ has its zeroes only at the points $\{t_i\}_1^m$, and each of them are simple. Hence

$$\operatorname{sign} B(\cdot, \theta) = \pm \operatorname{sign} \omega(\cdot).$$

It remains to investigate the sign of $B(x,\theta)$ for $x \to +\infty$. We have

$$2 \cdot B(x,\theta) = \frac{1}{(m-1)!}(x-\theta)_{\pm}^{m-1} - \frac{1}{(m-1)!}\sum_{i=1}^{m}\frac{\omega_i(x)}{\omega_i(t_i)}(t_i-\theta)_{\pm}^{m-1}$$

$$= \frac{1}{(m-1)!}(x-\theta)^{m-1} - \frac{1}{(m-1)!}r_{m-1,\theta}(x), \qquad x \to +\infty,$$

where by (4.2) the leading coefficient of the polynomial $r_{m-1,\theta}(x)$ is equal to

$$\sum_{i=1}^{m}\frac{(t_i-\theta)_{\pm}^{m-1}}{\omega_i(t_i)} = c(\theta) < 1.$$

Hence,

$$\operatorname{sign} B(x,\theta) = \operatorname{sign}\omega(x) > 0, \qquad x \to +\infty,$$

and the lemma is proved.

COROLLARY. *Let $\theta \in (t_1, t_m)$. Then*

$$\operatorname{sign} B_1(t_i,\theta) = \operatorname{sign}\omega_i(t_i), \qquad i = \overline{1,m}.$$

Proof. Since

$$B(t_i,\theta) = \omega(t_i) = 0$$

and by Lemma 7

$$\operatorname{sign} B(\cdot,\theta) = \operatorname{sign}\omega(\cdot),$$

we have only to make use of the definitions

$$B_1(t_i,\theta) = \frac{\partial}{\partial x}B(x,\theta)\bigg|_{x=t_i}, \qquad \omega_i(t_i) = \frac{\partial}{\partial x}\omega(x)\bigg|_{x=t_i}.$$

5. PROOF OF LEMMA 1

Let us recall the statements we are going to prove, preserving their enumeration from Section 3.

LEMMA 1. *If* $x \in J_{m,k}(\Delta)$ *and* $B_k(x,\theta_x) = 0$, *then*

$$\operatorname{sign} S_k(x,\theta_x) = \operatorname{sign} B_k'(x,\theta_x), \tag{3.15}$$

$$\operatorname{sign} S_k(x,\theta_x) = -\operatorname{sign} \omega_\Delta^{(k+1)}(x), \tag{3.16}$$

$$\operatorname{sign} S_k(x,\theta_x) = \operatorname{sign} B_{k-1}(x,\theta_x), \tag{3.17}$$

$$\operatorname{sign} B_{k-1}(x,\theta_x) = -\operatorname{sign} B_{k+1}(x,\theta_x), \tag{3.18}$$

and for $x = z$, *i.e., if* $S_{k+1}(x,\theta_x) = 0$, *moreover*,

$$\operatorname{sign} S_k(x,\theta_x) = -\operatorname{sign} S_{k+2}(x,\theta_x). \tag{3.19}$$

5.1. *Proof of Equality* (3.15). Since

$$S^{(m)}(x,\theta) = \frac{\partial^m}{\partial x^m} S(x,\theta) = \operatorname{sign}(x-\theta),$$

by Lemma B we have

$$S_k(x,\theta_x) = S^{(k)}(x,\theta_x) = -\int_a^{\theta_x} B_k(x,\theta)\,d\theta + \int_{\theta_x}^b B_k(x,\theta)\,d\theta.$$

By Lemma C, item (ii) the function $B_k(x,\cdot)$ on the interval (t_1,t_m) has the unique simple zero at the point θ_x, therefore,

$$\operatorname{sign} S_k(x,\theta_x) = \operatorname{sign} B_k(x,\theta_x+0) \tag{5.1}$$

$$= \operatorname{sign} B_k(x,t_m-0). \tag{5.2}$$

By the same arguments

$$\operatorname{sign} B_k'(x,\theta_x) = \operatorname{sign} B_k(x,\theta_x+0),$$

thus, by (5.1)

$$\operatorname{sign} S_k(x,\theta_x) = \operatorname{sign} B_k'(x,\theta_x).$$

5.2. *Proof of Equality* (3.16). Differentiating (2.5) k times with respect to x, we obtain

$$B_k(x,\theta) = \frac{1}{(m-1-k)!}(x-\theta)_+^{m-1-k} - \frac{1}{(m-1)!}\sum_{i=1}^m \frac{\omega_i^{(k)}(x)}{\omega_i(t_i)}(t_i-\theta)_+^{m-1}.$$

Hence we conclude that

$$B_k(x,\theta) = -\frac{c_m \omega_m^{(k)}(x)}{\omega_m(t_m)}(t_m - \theta)^{m-1}, \qquad \max(x, t_{m-1}) < \theta < t_m;$$

and, therefore,

$$\operatorname{sign} B_k(x, t_m - 0) = -\operatorname{sign} \omega_m^{(k)}(x) \cdot \operatorname{sign} \omega_m(t_m).$$

Obviously,

$$\operatorname{sign} \omega_m(t_m) > 0;$$

by Lemma A

$$\operatorname{sign} \omega_m^{(k)}(x) = \operatorname{sign} \omega_\Delta^{(k+1)}(x);$$

by (5.2)

$$\operatorname{sign} B_k(x, t_m - 0) = \operatorname{sign} S_k(x, \theta_x);$$

i.e.,

$$\operatorname{sign} S_k(x, \theta_x) = -\operatorname{sign} \omega_\Delta^{(k+1)}(x),$$

which was to be proved.

5.3. *Proof of Equality* (3.17). Differentiating both sides of (4.1) k times with respect to x and substituting $\theta = \theta_x$, with regards for the equality $B_k(x, \theta_x) = 0$, we obtain

$$S_k(x, \theta_x) = 2 \cdot \frac{k}{m} B_{k-1}(x, \theta_x) + c(\theta_x)\frac{1}{m!}\omega^{(k)}(x).$$

However, by virtue of (4.2)

$$|c(\theta_x)| < 1,$$

and by Lemma D

$$|S_k(x, \theta_x)| > \frac{1}{m!}|\omega^{(k)}(x)|.$$

Thus

$$\operatorname{sign} S_k(x, \theta_x) = \operatorname{sign} B_{k-1}(x, \theta_x),$$

and (3.17) is established.

5.4. *Proof of Equalities* (3.18)–(3.19). These follow from Lemmas 4–5.

Lemma 1 is completely proved.

6. Proof of Lemma 2

LEMMA 2. *Let $p_\theta(x) = p(x,\theta)$ be the function defined by equality*

$$B'(x,\theta) = -B_1(x,\theta) + \tfrac{1}{2}p(x,\theta). \tag{3.10}$$

If $x \in J_{m,k}(\Delta)$, then for any $\theta \in (t_1, t_m)$

$$\operatorname{sign} p_\theta^{(k)}(x) = \operatorname{sign} \omega_\Delta^{(k+1)}(x).$$

Proof of Lemma 2. From definitions (3.10) and (2.4) it follows that for any $\theta \in (t_1, t_m)$

$$\begin{aligned} p_\theta(x) &= 2 \cdot B'(x,\theta) + 2 \cdot B_1(x,\theta) \\ &= \frac{1}{(m-2)!}\sum_{i=1}^{m} \frac{\omega_i(x)}{\omega_i(t_i)}(t_i - \theta)_\pm^{m-2} \\ &\quad - \frac{1}{(m-1)!}\sum_{i=1}^{m} \frac{\omega_i'(x)}{\omega_i(t_i)}(t_i - \theta)_\pm^{m-1}, \end{aligned}$$

i.e., $p_\theta(x)$ is a polynomial of degree $m-1$ with respect to x. Moreover,

$$p_\theta(t_i) = 2 \cdot B'(t_i,\theta) + 2 \cdot B_1(t_i,\theta), \qquad i = \overline{1,m}.$$

However, by virtue of (2.3)–(2.4)

$$\begin{aligned} &-2 \cdot B'(x,\theta) \\ &\quad = -\frac{\partial}{\partial\theta} 2 \cdot B(x,\theta) \\ &\quad = \frac{1}{(m-2)!}(x-\theta)_\pm^{m-2} - l_{m-1,\Delta}\left(\frac{1}{(m-2)!}(\cdot - \theta)_\pm^{m-2}, x\right), \end{aligned}$$

whence

$$B'(t_i,\theta) = 0, \qquad i = \overline{1,m},$$

and, therefore,

$$p_\theta(t_i) = 2 \cdot B_1(t_i,\theta), \qquad i = \overline{1,m}.$$

By the Lagrange interpolation formula

$$p_\theta(x) = \sum_{i=1}^{m} \frac{\omega_i(x)}{\omega_i(t_i)} p_\theta(t_i) = 2 \cdot \sum_{i=1}^{m} \frac{\omega_i(x)}{\omega_i(t_i)} B_1(t_i, \theta),$$

hence,

$$p_\theta^{(k)}(x) = 2 \cdot \sum_{i=1}^{m} \frac{B_1(t_i, \theta)}{\omega_i(t_i)} \omega_i^{(k)}(x).$$

By the corollary of Lemma 7 for $\theta \in (t_1, t_m)$

$$\operatorname{sign} B_1(t_i, \theta) = \operatorname{sign} \omega_i(t_i), \qquad i = \overline{1, m}.$$

By Lemma A for $x \in J_{m,k}(\Delta)$

$$\operatorname{sign} \omega_i^{(k)}(x) = \operatorname{sign} \omega_\Delta^{(k+1)}(x), \qquad i = \overline{1, m}.$$

Thus, for $x \in J_{m,k}(\Delta)$ and $\theta \in (t_1, t_m)$

$$\operatorname{sign} p_\theta^{(k)}(x) = \operatorname{sign} \omega_\Delta^{(k+1)}(x),$$

and Lemma 2 is proved.

7. Proof of Lemma 3

Lemma 3. *Let $q_\theta(x) = q(x, \theta)$ be the function defined by equality*

$$S_1(x, \theta) = 2 \cdot B(x, \theta) + q(x, \theta). \tag{3.11}$$

If $\theta \in (t_1, t_m)$ and $q_\theta^{(k)}(x) = 0$, then

$$\operatorname{sign} q_\theta^{(k+1)}(x) = -\operatorname{sign} \omega_\Delta^{(k+1)}(x).$$

Proof of Lemma 3. From Eqs. (3.11), (2.2), and (2.4) it follows that for any $\theta \in \mathbb{R}$

$$\begin{aligned} q_\theta(x) &= S_1(x, \theta) - 2 \cdot B(x, \theta) \\ &= \frac{1}{(m-1)!} \sum_{i=1}^{m} \frac{\omega_i(x)}{\omega_i(t_i)} (t_i - \theta)_\pm^{m-1} - \frac{1}{m!} \sum_{i=1}^{m} \frac{\omega_i'(x)}{\omega_i(t_i)} (t_i - \theta)_\pm^{m}, \end{aligned}$$

i.e., $q_\theta(x)$ is a polynomial of degree $m - 1$ with respect to x with leading coefficient equal to

$$\frac{1}{(m-1)!}c(\theta) = \frac{1}{(m-1)!}\sum_{i=1}^{m}\frac{(t_i-\theta)_{\pm}^{m-1}}{\omega_i(t_i)}.$$

We need two lemmas. The first of them is due to V. A. Markov, and the other will be proved in the next section.

LEMMA 8 [5, Lemma 2]. *Let*

$$r(x) = \prod_{j=1}^{s}(x-\mu_j), \qquad \mu_1 < \mu_2 < \cdots < \mu_s,$$

$$r_j(x) = r(x)/(x-\mu_j).$$

If

$$r^{(l)}(\xi) = 0,$$

then

$$\operatorname{sign} r_j^{(l)}(\xi) = \operatorname{sign} r^{(l+1)}(\xi), \qquad j = \overline{1,s}.$$

LEMMA 9. *For any $\theta \in (t_1, t_m)$ zeroes of the polynomials $q_\theta(x)$ and $\omega'_\Delta(x)$ interlace. Moreover, if*

$$q_\theta(x) = \frac{1}{(m-1)!}c(\theta)\prod_j(x-\delta_j(\theta)),$$

$$\delta_1(\theta) < \delta_2(\theta) < \cdots < \delta_{m-1}(\theta),$$

$$\omega'_\Delta(x) = m\prod_j(x-\tau_j), \qquad \tau_1 < \tau_2 < \cdots < \tau_{m-1},$$

then

$$\delta_1(\theta) < \tau_1 < \delta_2(\theta) < \tau_2 < \cdots < \delta_{m-1}(\theta) < \tau_{m-1}, \qquad c(\theta) \geq 0, \tag{7.1}$$

$$\tau_1 < \delta_1(\theta) < \tau_2 < \delta_2(\theta) < \cdots < \tau_{m-1} < \delta_{m-1}(\theta), \qquad c(\theta) \leq 0. \tag{7.2}$$

COROLLARY. *For any $\theta \in (t_1, t_m)$*

$$\operatorname{sign}\omega'_\Delta(\delta_j) = -\operatorname{sign} q'_\theta(\delta_j).$$

Having these statements, we can now prove Lemma 3 repeating arguments from V. A. Markov [5, Lemma 3].

By the Lagrange interpolation formula

$$\omega'_\Delta(x) = \sum_{j=1}^{m} \frac{\omega'_\Delta(\delta_j)}{q'_\theta(\delta_j)} q_{\theta,j}(x) + c q_\theta(x),$$

where

$$q_{\theta,j}(x) = q_\theta(x)/(x - \delta_j),$$

and if

$$q_\theta^{(k)}(x) = 0,$$

then

$$\omega_\Delta^{(k+1)}(x) = \sum_{j=1}^{m} \frac{\omega'_\Delta(\delta_j)}{q'_\theta(\delta_j)} q_{\theta,j}^{(k)}(x).$$

By Lemma 8

$$\operatorname{sign} q_{\theta,j}^{(k)}(x) = \operatorname{sign} q_\theta^{(k+1)}(x).$$

By corollary of Lemma 9

$$\operatorname{sign} \omega'_\Delta(\delta_j) = -\operatorname{sign} q'_\theta(\delta_j).$$

Hence

$$\operatorname{sign} \omega_\Delta^{(k+1)}(x) = -\operatorname{sign} q_\theta^{(k+1)}(x),$$

which was stated in Lemma 3.

8. Proof of Lemma 9

We will prove the following equivalent statement.

Lemma 9'. *Let $q_\theta(x) = q(x, \theta)$ be the function defined by equality*

$$S_1(x,\theta) = 2 \cdot B(x,\theta) + q(x,\theta) \tag{3.11}$$

and let

$$\omega'_\Delta(x) = m \prod_j (x - \tau_j), \qquad \tau_1 < \tau_2 < \cdots < \tau_{m-1}.$$

Then

$$\operatorname{sign} q_\theta(\tau_j) = \operatorname{sign} q(\tau_j, \theta) = (-1)^{m-1-j}, \qquad \theta \in (t_1, t_m). \tag{8.1}$$

Equalities (8.1) mean that zeroes of the polynomials $q_\theta(x)$ and $\omega'(x)$ interlace. To obtain relations (7.1)–(7.2) we must only take into account the sign of the leading coefficient of the polynomial $q_\theta \in P_{m-1}$.

Proof of Lemma 9'. Let

$$\omega'_\Delta(\xi) = 0, \qquad \xi \in \{\tau_1, \tau_2, \ldots, \tau_{m-1}\}.$$

Let us calculate zeroes of the function $q_x(\cdot) \equiv q(x, \cdot)$ on the interval $[t_1, t_m]$ for $x = \xi$.

By (3.3) and item (iii) of Lemma 3

$$\frac{\partial^l}{\partial\theta^l} q(x,\theta)\Big|_{\theta=t_1,t_m} = \frac{\partial^l}{\partial\theta^l} S_1(x,\theta)\Big|_{\theta=t_1,t_m} - 2\cdot\frac{\partial^l}{\partial\theta^l} B(x,\theta)\Big|_{\theta=t_1,t_m}$$

$$= \begin{cases} S_1(x,\theta)|_{\theta=t_1,t_m}, & l=0,\\ 0, & 1 \le l \le m-2.\end{cases}$$

Differentiating both sides of (4.1) with respect to x, we find

$$S_1(x,\theta) = 2\cdot\frac{1}{m}(x-\theta)B_1(x,\theta) + 2\cdot\frac{1}{m}B(x,\theta) + c(\theta)\frac{1}{m!}\omega'(x),$$

whence, using again the finiteness of $B_l(x,\cdot)$, with the aid of (4.2) we obtain

$$S_1(x,\theta)|_{\theta=t_1,t_m} = c(\theta)\frac{1}{m!}\omega'(x)\Big|_{\theta=t_1,t_m}$$

$$= \begin{cases} \dfrac{1}{m!}\omega'(x), & \theta = t_1,\\[2mm] -\dfrac{1}{m!}\omega'(x), & \theta = t_m.\end{cases}$$

Thus,

$$q_\xi^{(l)}(\theta)|_{\theta=t_1,t_m} = \frac{\partial^l}{\partial\theta^l} q(\xi,\theta)\Big|_{\theta=t_1,t_m} = 0, \qquad l = \overline{0, m-2}. \tag{8.2}$$

On the other hand, by (3.11) and (3.3)

$$q'_x(\theta) = -2\cdot B_1(x,\theta) - 2\cdot B'(x,\theta),$$

and using for $B(x, \theta)$ representation (2.5) we obtain

$$\frac{1}{2}q'_x(\theta) = \frac{1}{(m-1)!}\sum_{i=1}^{m}\frac{\omega'_i(x)}{\omega_i(t_i)}(t_i-\theta)_+^{m-1} - \frac{1}{(m-2)!}\sum_{i=1}^{m}\frac{\omega_i(x)}{\omega_i(t_i)}(t_i-\theta)_+^{m-2}.$$

From this relation by differentiating $m-2$ times with respect to θ, taking into account the equalities

$$\frac{\partial}{\partial\theta}(t_i-\theta)_+^l = (-l)\cdot(t_i-\theta)_+^{l-1}, \qquad (t_i-\theta)_+^l = (t_i-\theta)_+^{l-1}\cdot(t_i-\theta)$$

we derive

$$\begin{aligned}
(-1)^{m-2}&\frac{1}{2}q_x^{(m-1)}(\theta)\\
&= \sum_{i=1}^{m}\frac{\omega'_i(x)\cdot(t_i-\theta)}{\omega_i(t_i)}(t_i-\theta)_+^0 - \sum_{i=1}^{m}\frac{\omega_i(x)}{\omega_i(t_i)}(t_i-\theta)_+^0\\
&= \sum_{i=1}^{m}\frac{\omega'_i(x)\cdot(t_i-x)}{\omega_i(t_i)}(t_i-\theta)_+^0 - \sum_{i=1}^{m}\frac{\omega_i(x)}{\omega_i(t_i)}(t_i-\theta)_+^0\\
&\quad + (x-\theta)\sum_{i=1}^{m}\frac{\omega'_i(x)}{\omega_i(t_i)}(t_i-\theta)_+^0.
\end{aligned}$$

Since

$$\omega(x) = (x-t_i)\cdot\omega_i(x),$$

we have

$$\omega'(x) = \omega_i(x) + (x-t_i)\cdot\omega'_i(x),$$

whence

$$(t_i-\xi)\cdot\omega'_i(\xi) = \omega_i(\xi).$$

Therefore,

$$q_\xi^{(m-1)}(\theta) = 2\cdot(-1)^{m-2}(\xi-\theta)\sum_{i=1}^{m}\frac{\omega'_i(\xi)}{\omega_i(t_i)}(t_i-\theta)_+^0. \qquad (8.3)$$

From this formula it is seen that on the interval $[t_1, t_m]$ the function $q_\xi^{(m-1)}(\cdot)$ can change its sign only at the points ξ and $\{t_i\}_2^{m-1}$, i.e.,

$$\nu\left[q_\xi^{(m-1)}(\cdot)\right] \leq m - 1.$$

However, by virtue of (8.2) the function $q_\xi(\cdot)$ has at least $2(m-1)$ zeroes counting multiplicity, i.e.,

$$\nu\left[q_\xi(\cdot)\right] \geq 2(m-1).$$

Hence, with the aid of Rolle's Theorem we conclude that

$$\nu\left[q_\xi(\cdot)\right] = 2(m-1)$$

i.e., the function $q_\xi(\theta) = q(\xi, \theta)$ has no zeroes, different from (8.2), and, therefore, for each $\xi \in \{\tau_j\}_1^{m-1}$

$$\operatorname{sign} q_\xi(\theta) = \operatorname{const}(\xi), \qquad \theta \in (t_1, t_m).$$

It remains to investigate the sign of this constant for $\xi = \tau_j$. By (8.2) for any $\theta \in (t_1, t_m)$

$$\operatorname{sign} q_\xi(\theta) = \operatorname{sign} q_\xi(t_m - 0) = (-1)^{m-1} \operatorname{sign} q_\xi^{(m-1)}(t_m - 0),$$

and by (8.3)

$$\operatorname{sign} q_\xi^{(m-1)}(t_m - 0) = (-1)^{m-1} \operatorname{sign} \omega_m'(\xi) \operatorname{sign} \omega_m(t_m).$$

Thus,

$$\operatorname{sign} q_\xi(\theta) = \operatorname{sign} \omega_m'(\xi).$$

By Lemma 8

$$\operatorname{sign} \omega_m'(\xi) = \operatorname{sign} \omega_\Delta''(\xi)$$

and it is evident that if

$$\omega_\Delta'(x) = m \prod_j (x - \tau_j), \qquad \tau_1 < \tau_2 < \cdots < \tau_{m-1},$$

then

$$\operatorname{sign} \omega_\Delta''(\tau_j) = (-1)^{m-1-j}.$$

Finally,

$$\operatorname{sign} q_\theta(\tau_j) = \operatorname{sign} q_{\tau_j}(\theta) = (-1)^{m-1-j}, \qquad \theta \in (t_1, t_m),$$

and relations (8.1) and thus Lemma 9 are proved.

9. The Case $k = m - 1$

For the sake of completeness let us briefly reproduce from [6] the proof of Theorem 2 for the case $k = m - 1$.

In this case, due to the definition (2.5)

$$B_{m-1}(x,\theta) = (x-\theta)_+^0 - \sum_{i=1}^{m} \frac{1}{\omega_i(t_i)}(t_i - \theta)_+^{m-1}.$$

Consider the classical B-spline $b(t)$ of degree $m-2$ with the breakpoints $\{t_i\}_{i=1}^m$

$$b(t) = (m-1)\sum_{i=1}^{m} \frac{(t_i - t)_+^{m-2}}{\omega'_\Delta(t_i)} \tag{9.1}$$

which has the properties

$$\operatorname{supp} b(\cdot) = (t_1, t_m), \qquad b(\cdot) \geq 0, \qquad \int_{t_1}^{t_m} b(t)\,dt = 1.$$

It is seen that

$$B_{m-1}(x,\theta) = \int_{-\infty}^{\theta} [b(t) - \delta(x-\theta)]\,dt = \int_{+\infty}^{\theta} [b(t) - \delta(x-\theta)]\,dt,$$

where δ is the Dirac function.

Thus, the kernel $B_{m-1}(x,\theta)$ as a function with respect to θ changes its sign at the point

$$\theta_x = x, \qquad x \in J_{m,k}(\Delta) = (t_1, t_m),$$

and for the value of the pointwise deviation $L_{m,m-1}(\Delta, x)$, we obtain by Lemma D and Eq. (2.2) the following expression

$$\begin{aligned} L_{m,m-1}(\Delta, x) &= |S_{m-1}(x,\theta_x)| = |S_{m-1}(x,x)| \\ &= -S_{m-1}(x,x) = \frac{1}{m}\sum_{i=1}^{m} \frac{(t_i - x)_\pm^m}{\omega_i(t_i)}, \end{aligned} \tag{9.2}$$

where as before $z_\pm^m = z^m \cdot \operatorname{sign} z$.

With the aid of (9.1), (9.2) it can be readily verified that

$$L_{m,m-1}(\Delta, x) = \int_{t_1}^{x}\int_{t_1}^{\theta} b(t)\,dt\,d\theta + \int_{t_m}^{x}\int_{t_m}^{\theta} b(t)\,dt\,d\theta,$$

and thus

$$L''_{m,m-1}(\Delta, x) = 2b(x) \geq 0$$

which completes the proof.

Remark. The case $k = m - 1$ of Theorem 2 given in Section 9 is also contained in the recent paper by G. Howell [*J. Approx. Theory* **67** (1991), 164–173].

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